A Novel MUSIC Algorithm for Direction-of-Arrival Estimation without the Estimate of Covariance Matrix and Its Eigendecomposition

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Abstract: In this paper, a new MUSIC algorithm for direction of arrival (DOA) estimation is developed, based on the multi-stage wiener filter (MSWF). Unlike the classical MUSIC algorithm, the proposed method only involves the forward recursions of the MSWF to find the noise subspace even in the case of coherent signals, dose not require the estimate of an array covariance matrix or its eigendecomposition. Therefore, the proposed method is computationally advantageous over the classical MUSIC algorithm that resorts to computing the sample covariance matrix and its eigenvectors. The performance of the proposed method is demonstrated by numerical results.

Keywords: Multi-stage wiener filter, array signal processing, direction of arrival, MUSIC.

I. Introduction

It is interesting to estimate the direction of arrival (DOA) parameters of signals in the noisy background in such areas as communication, radar, sonar and geophysical seismology [1] [2]. The well known subspace based methods that dependent on the decomposition of the observation space into signal subspace and noise subspace, can provide high-resolution DOA estimations with good estimation accuracy. However, the classical subspace based methods such as the MUSIC-type [3] methods, involve the estimate of the covariance matrix and its eigendecomposition. As a result, the classical subspace based methods are rather computationally intensive, especially for the case where the model orders in these matrices are large. Recently, the methods called reduced-order correlation kernel estimation technique (ROCKET) [4] and ROCK MUSIC algorithm [5] were presented to high-resolution spectral estimation which dose not need the inverse of the covariance matrix. Nevertheless, the ROCK MUSIC technique still needs the forward and backward recursions of the multi-stage wiener filter (MSWF) [6], which increase the computational complexity of the algorithm. Moreover, the ROCKET algorithm involves complex matrix-matrix products to find the reduced-rank data matrix and the reduced-rank autoregressive (AR) weight vector. This implies that additionally computational cost is included.

In this paper, we present a low computational complexity MUSIC method for DOA estimation, based on the MSWF. Unlike the ROCK MUSIC [5] technique, the proposed method merely involves the forward recursions of the MSWF to extract the noise subspace even in the case of coherent signals, does not need the backward recursion of the MSWF or any complex matrix-matrix products, thereby further reducing the computational complexity of the algorithm. Compared to the classical eigendecomposition based methods, the proposed method avoids the estimate of the covariance matrix and its eigendecomposition. Thus, the presented method is computationally efficient. Basically, the proposed method works similarly to the classical MUSIC method but finds the noise subspace in a more computationally efficient way, which is the distinguishing feature of the proposed method.

II. Problem Formulation

A. Data Model

Let us consider a uniform linear array (ULA) composed of M isotropic sensors. Assume that P narrow-band signals impinge upon the ULA from distinct directions $\theta_1, \theta_2, \cdots, \theta_P$. The $M \times 1$ output vector of the array, which is corrupted by additive noise, at the kth snapshot can be expressed as

$$\mathbf{x}(k) = \sum_{i=1}^{P} \mathbf{a}(\theta_i) s_i(k) + \mathbf{n}(k) \quad k = 0, \cdots, N-1 \qquad (1)$$

where $s_i(k)$ is the scalar complex waveform referred to as the ith signal, $\mathbf{n}(k) \in \mathcal{C}^{M \times 1}$ is the complex noise vector, N and P denote the number of snapshots and the number of signals, respectively, $\mathbf{a}(\theta_i)$ is the steering vector of the array toward direction θ_i and takes the following form

$$\mathbf{a}(\theta_i) = \frac{1}{\sqrt{M}} \left[1, e^{j\varphi_i}, \cdots, e^{j(M-1)\varphi_i}\right]^T \qquad (2)$$

where $\varphi_i = \frac{2\pi d}{\lambda} \sin\theta_i$ in which $\theta_i \in (-\pi/2, \pi/2)$, d and λ are inter-element spacing and the wavelength, respectively, and the superscript $(\cdot)^T$ denotes the transpose operator.

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In matrix form, Equation (1) becomes

$$\mathbf{x}(k) = \mathbf{A}(\theta)\mathbf{s}(k) + \mathbf{n}(k) \quad k = 0, 1, \cdots, N-1 \tag{3}$$

where

$$\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_P)] \tag{4}$$

$$\mathbf{s}(k) = [s_1(k), s_2(k), \cdots, s_P(k)]^T \tag{5}$$

are the $M \times P$ steering matrix and the $P \times 1$ complex signal vector, respectively. Throughout the paper we assume that $M > P$. Furthermore, the background noise uncorrelated with the signals is modeled as a stationary, spatially-temporally white, zero-mean, Gaussian complex random process.

B. Multi-Stage Wiener Filter

It is well known that the wiener filter (WF) $\mathbf{w}_{wf} \in \mathcal{C}^{M\times 1}$ can be used to estimate the desired signal $d(k) \in \mathcal{C}$ from the observation data $\mathbf{x}(k)$ in the minimum mean square error (MMSE) sense. Thereby, we get the following design criterion

$$\mathbf{w}_{wf} = \arg\min_{\mathbf{w}} E\{|d(k) - \mathbf{w}^H\mathbf{x}(k)|^2\} \tag{6}$$

where $\hat{d}(k) = \mathbf{w}^H\mathbf{x}(k)$ represents the estimate of the desired signal $d(k)$, and $\mathbf{w} \in \mathcal{C}^{M\times 1}$ is the linear filter. Solving (6) leads to the Wiener-Hopf equation

$$\mathbf{R_x}\mathbf{w}_{wf} = \mathbf{r}_{\mathbf{x}d} \tag{7}$$

where $\mathbf{R_x} = E[\mathbf{x}(k)\mathbf{x}^H(k)]$, $\mathbf{r}_{\mathbf{x}d} = E[\mathbf{x}(k)d^*(k)]$. The classical wiener filter, *i.e.*, $\mathbf{w}_{wf} = \mathbf{R}_{\mathbf{x}}^{-1}\mathbf{r}_{\mathbf{x}d}$, is computationally intensive for large M since the inverse of the covariance matrix is involved. The MSWF developed by Goldstein *et al* [6] is to find an approximate solution to the Wiener-Hopf equation which does not need the inverse or eigendecomposition of the covariance matrix. In contrast to the *principal components* (PC) method [7] and the *cross-spectral* (CS) metric [8], the MSWF requires much lower computational cost, offers faster convergence and can work in the low-sample support operational environment where other adaptive algorithms fail. The MSWF based on the data-level lattice structure [9] is given as follows:

- *Initialization*: $d_0(k)$ and $\mathbf{x}_0(k) = \mathbf{x}(k)$.
- *Forward Recursion*: For $i = 1, 2, \cdots, D$:
  $\mathbf{h}_i = E[\mathbf{x}_{i-1}(k)d^*_{i-1}(k)]/\|E[\mathbf{x}_{i-1}(k)d^*_{i-1}(k)]\|_2$;
  $d_i(k) = \mathbf{h}_i^H\mathbf{x}_{i-1}(k)$;
  $\mathbf{x}_i(k) = \mathbf{x}_{i-1}(k) - \mathbf{h}_i d_i(k)$.
- *Backward Recursion*: For $i = D, D-1, \cdots, 1$ with $e_D(k) = d_D(k)$:
  $w_i = E[d_{i-1}(k)e_i^*(k)]/E[|e_i(k)|^2]$;
  $e_{i-1}(k) = d_{i-1}(k) - w_i^* e_i(k)$.

The corresponding block diagram can be found in Fig. 1. In the algorithm above, the reference signal $d_0(k)$ can be acquired from the training data or the spreading codes of users in blind mode. The pre-filtering matrix $\mathbf{T}_M = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_M]$ is obtained by performing $M(D = M)$ forward recursions of the MSWF.

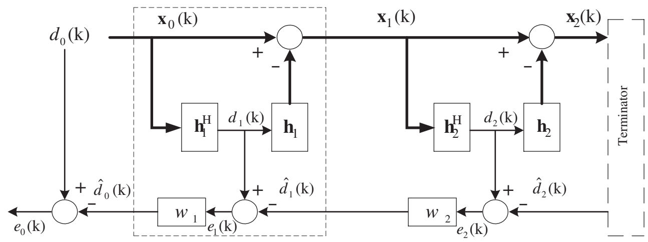

Fig. 1. Lattice structure of the MSWF. The dashed line denotes the basic box for each additional stage.

III. A Novel MUSIC Estimator

It is shown in [10] that all the matched filters $\mathbf{h}_i$, $i = 1, 2, \cdots, D(D \leq P)$ are contained in the column space of $\mathbf{A}(\theta)$. It follows that the orthogonal matched filters $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_P$ span the signal subspace, namely

$$span\{\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_P\} = col\{\mathbf{A}(\theta)\}. \tag{8}$$

Since all the matched filters $\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_M$ are orthogonal for the special choice of the blocking matrix $\mathbf{B}_i = \mathbf{I} - \mathbf{h}_i\mathbf{h}_i^H$, the matched filters after the P stage of the MSWF are orthogonal to the signal subspace, $i.e.$, $\mathbf{h}_i \perp col\{\mathbf{A}(\theta)\}$ for $i = P+1, P+2, \cdots, M$. Therefore, the last $M-P$ matched filters span the orthogonal complement of the signal subspace, namely the noise subspace

$$span\{\mathbf{h}_{P+1}, \mathbf{h}_{P+2}, \cdots, \mathbf{h}_M\} = null\{\mathbf{A}(\theta)\}. \tag{9}$$

Equation (9) indicates that the noise subspace can be readily obtained by performing the forward recursions of the MSWF, and thus the MUSIC estimator based on the noise subspace can be exploited to produce peaks at the DOA locations. However, for coherent signals, the noise subspace estimated by this method is incorrect anymore. That is to say, the last $M-P$ matched filters do not span a noise subspace for the case where the signals are coherent. As a result, we must resort to the smoothing techniques to decorrelate the coherent signals.

For the spatial smoothing technique [11], an array consisting of M sensors is subdivided into L subarrays. Thereby, the number of elements per subarray is $M_L = M - L + 1$. For $l = 1, 2, \cdots, L$, let the $M_L \times M$ matrix $\mathbf{J}_l$ be a selection matrix, which takes the following form

$$\mathbf{J}_l = \left[\ \mathbf{0}_{M_L\times(l-1)} \ \vdots \ \mathbf{I}_{M_L\times M_L} \ \vdots \ \mathbf{0}_{M_L\times(M-l-M_L+1)}\ \right]. \tag{10}$$

The selection matrix $\mathbf{J}_l$ is exploited to select part of the $M \times N$ observation data matrix $\mathbf{X}_0 = [\mathbf{x}_0(0), \mathbf{x}_0(1), \cdots, \mathbf{x}_0(N-1)]$, which associates with the lth subarray. Hence, the spatially smoothed $M_L \times LN$ data matrix $\bar{\mathbf{X}}_0$ is constructed by

$$\bar{\mathbf{X}}_0 = [\mathbf{J}_1\mathbf{X}_0 \ \ \mathbf{J}_2\mathbf{X}_0 \ \ \cdots \ \ \mathbf{J}_L\mathbf{X}_0] \in \mathcal{C}^{M_L\times LN}. \tag{11}$$

Similarly to the spatially smoothed data matrix $\bar{\mathbf{X}}_0$, the "spatially smoothed" training signal vector should have the following form

$$\bar{\mathbf{d}}_0 = [\underbrace{\mathbf{d}_0;\ \mathbf{d}_0;\ \cdots;\ \mathbf{d}_0}_{L}] \in \mathcal{C}^{LN\times 1} \tag{12}$$

where $\mathbf{d}_0 = [d_0(0), d_0(1), \cdots, d_0(N-1)]^T \in \mathcal{C}^{N\times 1}$. Thus, the ith spatially smoothed matched filter of the MSWF is given by

$$\bar{\mathbf{h}}_i = \frac{\bar{\mathbf{r}}_{\mathbf{x}_{i-1}d_{i-1}}}{\|\bar{\mathbf{r}}_{\mathbf{x}_{i-1}d_{i-1}}\|_2} = \frac{\bar{\mathbf{X}}_{i-1}\mathbf{d}^*_{i-1}}{\|\bar{\mathbf{X}}_{i-1}\mathbf{d}^*_{i-1}\|_2}. \tag{13}$$

Hence, the low-complexity MUSIC algorithm for direction finding is summed as follows:

Step1: Perform the spatial smoothing technique to the $M \times N$ data matrix $\mathbf{X}_0$, obtain the spatially smoothed $M_L \times NL$ data matrix $\bar{\mathbf{X}}_0$.

Step2: Construct the spatially smoothed training sequence vector $\bar{\mathbf{d}}_0$ by the way shown in (12).

Step3: Perform the following set of recursions

For $i = 1, 2, \cdots, M_L$:
$\bar{\mathbf{h}}_i = \bar{\mathbf{X}}_{i-1}\bar{\mathbf{d}}^*_{i-1}/\|\bar{\mathbf{X}}_{i-1}\bar{\mathbf{d}}^*_{i-1}\|_2$,
$\bar{\mathbf{d}}_i = \bar{\mathbf{h}}_i^H\bar{\mathbf{X}}_{i-1}$,
$\bar{\mathbf{X}}_i = \bar{\mathbf{X}}_{i-1} - \bar{\mathbf{h}}_i\bar{\mathbf{d}}_i$.

Obtain the noise subspace $\bar{\mathbf{N}}_{M_L-P} = \left[\bar{\mathbf{h}}_{P+1}, \bar{\mathbf{h}}_{P+2}, \cdots, \bar{\mathbf{h}}_{M_L}\right]$.

Step4: Exploit the MUSIC estimator $P_{MUSIC}(\theta) = \frac{1}{\mathbf{a}^H_{M_L}(\theta)\mathbf{N}_{M_L-P}\mathbf{N}^H_{M_L-P}\mathbf{a}_{M_L}(\theta)}$ to produce peaks at the DOA locations, where $\mathbf{a}_{M_L}(\theta) = \frac{1}{\sqrt{M_L}}[1, e^{j\varphi_i}, \cdots, e^{j(M_L-1)\varphi_i}]^T$. Alternatively, the DOAs can also be estimated by the root-MUSIC algorithm: finding the P roots, say $\hat{z}_1,\hat{z}_2,\cdots,\hat{z}_P$ that have the largest magnitude, of the root-MUSIC polynomial $D(z) = z^{M_L-1}\mathbf{p}^T(z^{-1})\bar{\mathbf{N}}_{M_L-P}\bar{\mathbf{N}}^H_{M_L-P}\mathbf{p}(z)$ where $\mathbf{p}(z) = \left[1, z. \cdots, z^{M_L-1}\right]^T$, yields the DOA estimates as $\hat{\theta}_i = \arcsin\left(\frac{\lambda \arg(\hat{z}_i)}{2\pi d}\right)$ in which $\arg(\hat{z}_i)$ denotes the phase angle of the complex number $\hat{z}_i$.

Remark: Notice that the low-complexity MUSIC algorithm given above avoids the formation of blocking matrices, and all the operations only involve complex matrix-vector products, thereby requiring the computational complexity of $O(M_L NL)$ for each matched filter $\mathbf{h}_i, i \in \{1, 2, \cdots, M_L\}$. To fulfil the estimation of the noise subspace, M_L stages of the MSWF are needed. Thus, the computational cost of the proposed method is only $O(M_L^2 NL)$ flops. However, the classical MUSIC method includes the estimate of the spatially smoothed covariance matrix and its eigendecomposition, which require $O(M_L^2 NL + M_L^3)$ flops. Therefore, the proposed method is computationally attractive.

IV. Numerical Results

We consider the case where there are three signals impinging upon the ULA consisting of 14 sensors from the same signal source. The first is a direct-path signal and the others refer to the scaled and delayed replicas of the first signal that represent the multipaths or the "smart" jammers. The propagation constants are $\{1, -0.8 + j0.3, 0.4 - j0.6\}$. We assume that the true DOAs are $\{-10^{\circ}, 0^{\circ}, 5^{\circ}\}$ and the number of signals is known a priori. The background noise is a stationary Gaussian white random process.

The spatial spectra of the proposed method and the classical MUSIC algorithm are shown in Fig. 2, where the number of snapshots is 256 and the signal to noise ratio (SNR) is 18dB. SNR is defined as $10\log(\sigma^2_{\mathbf{s}}/\sigma^2_{\mathbf{n}})$, where $\sigma^2_{\mathbf{s}}$ is the power of each signal in single sensor. It can be observed that the proposed estimator works very well like the classical MUSIC estimator. Fig. 3 shows the root-mean-square error (RMSE) of the estimated DOA for the first signal versus SNR, based on 500 Monte Carlo runs. The number of snapshots is equal to 128. It is shown in Fig. 3 that the proposed MUSIC estimator clearly outperforms the classical SS-MUSIC algorithm when SNR$\leq$ 18dB, and provides the same estimation accuracy as the latter when SNR$>$ 18dB. As SNR increases, the RMSE's of the two methods approach to the Cramér-Rao bound (CRB). The RMSE's of the estimated DOA of the first signal for the two methods versus the number of snapshots are demonstrated in Fig. 4, where SNR=18dB. It can be observed that the proposed method surpasses the classical MUSIC estimator over the range of the number of snapshots that we simulated.

V. Conclusion

We have presented a low-complexity MUSIC algorithm for DOA estimation in this paper. The proposed method only requires $O(M_L^2 NL)$ flops that are equivalent to the computational complexity of estimating the spatially smoothed covariance matrix. In contrast to the classical MUSIC method which involves the estimate of the spatially smoothed covariance matrix and its eigendecomposition, thereby requiring the computational cost of $O(M_L^2 NL + M_L^3)$, the proposed method is more computationally efficient. Numerical results indicate that the proposed method outperforms the classical MUSIC method in estimation accuracy.

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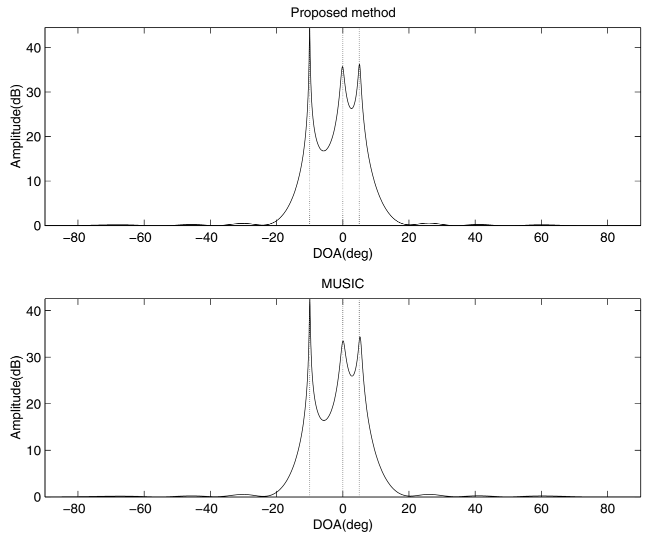

Fig. 2. Spatial spectra of the proposed method and the classical MUSIC method. One trial.

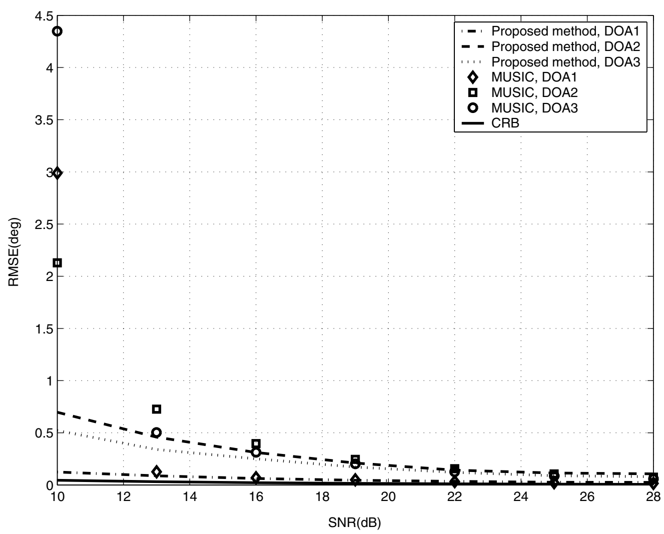

Fig. 3. RMSE of estimated DOA for signal 1 versus SNR. The number of snapshots and the number of sensors are equal to 128 and 14, respectively. 500 trials.

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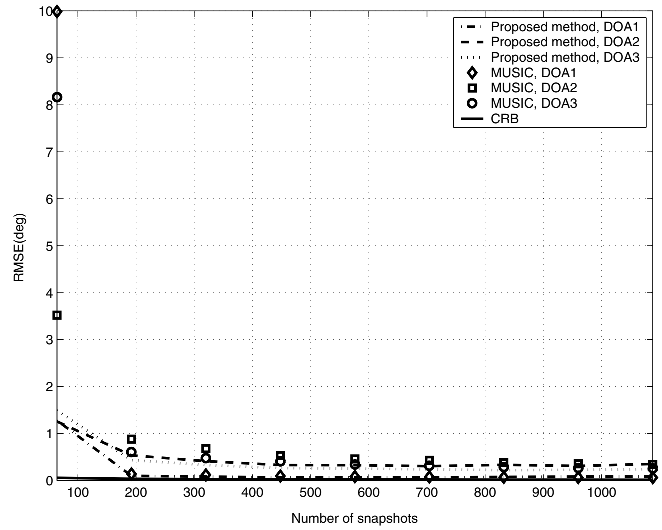

Fig. 4. RMSE of estimated DOA for signal 1 versus number of snapshots. The number of sensors and SNR are equal to 14 and 18dB, respectively. 500 trials.

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