# A New Method for Noise Subspace Estimation Based on The Spatial Smoothing Lanczos Algorithm

Lei Huang, Linrang Zhang and Shunjun Wu
Key Lab for Radar Signal Processing, Xidian University, Xi' an, Shaanxi 710071, P. R. China

**Abstract**

A new method is proposed to estimate a noise subspace. It is shown that the redundant pre-filters of the multistage wiener filter (MSWF) are capable of creating an orthogonal basis for the noise subspace. Based on the classical spatial smoothing technique and the *Lanczos* algorithm, a novel technique is presented to obtained the noise subspace in the case of coherent signals. The new estimator outperforms its counterparts in terms of computational complexity. Finally, the theoretical observations is illustrated by numerical results.

## I. Introduction

It is shown that the classical MUSIC algorithm suffers from high computational load in the case of large number of sensors, mainly due to the fact that it needs to compute the eigenvectors associated with the covariance matrix. On the other hand, the MUSIC estimator fails to form peaks at the true direction-of-arrival (DOA) locations of signals when coherent signals exist since the signal source covariance matrix is singular in this case.

This paper focus on a fast algorithm of the noise subspace estimation based on the spatial smoothing *Lanczos* method. It is shown in what follows that a noise subspace can be obtained with low computational cost and simple structure, *i.e.*, only the forward recursion of the multi-stage wiener filter (MSWF) recently presented by Goldstein *et al* [1] is needed.

## II. Data Model

We consider the scenario of an $M$-element uniform linear array (ULA), with $K$ statistically independent narrowband signals impinging upon the array in different directions. All the signal sources are assumed to undergo multipath propagation, producing a set of delayed and scaled replicas of itself. In the sequel, the number of paths from the $i$th transmitter to the receiver is denoted by $p_i$. Accordingly, there are $P = \sum_{k=1}^{K} p_k (P < M)$ wavefronts impinging on the array. The received signals corrupted by additive noise can be written as

$$\mathbf{x}_0(i) = \mathbf{A}(\theta)\mathbf{s}(i) + \mathbf{n}(i) \quad i = 0, 1, \cdots, N-1 \tag{1}$$

We define the following matrices and vectors

$$\begin{aligned}
\mathbf{A}(\theta) &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_K \end{bmatrix} \\
\mathbf{A}_k &= \begin{bmatrix} \mathbf{a}(\theta_{k,1}) & \mathbf{a}(\theta_{k,2}) & \cdots & \mathbf{a}(\theta_{k,p_k}) \end{bmatrix} \\
\mathbf{s}(i) &= \begin{bmatrix} \mathbf{s}_1(i) & \mathbf{s}_2(i) & \cdots & \mathbf{s}_K(i) \end{bmatrix} \\
\mathbf{s}_k(i) &= \begin{bmatrix} c_{k,1} & c_{k,2} & \cdots & c_{k,p_k} \end{bmatrix} u_k(i)
\end{aligned}$$

It is shown from Equation (1) that the received signals can be rewritten in terms of the independent sources as [2]

$$\mathbf{x}_0(i) = \mathbf{A}(\theta)\mathbf{C}\mathbf{u}(i) + \mathbf{n}(i) \tag{2}$$

where$\mathbf{C} = diag\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_K\}$, $\mathbf{c}_k = [c_{k,1}, c_{k,2}, \cdots, c_{k,p_k}]^T$ and $\mathbf{u}(i) = [u_1(i), u_2(i), \cdots, u_K(i)]^T$ are the corrupt matrix, the corrupt vector and the signal vector, respectively. The noise vector $\mathbf{n}(i) \in \mathcal{C}^{M\times 1}$ is assumed to be a stationary Gaussian white random process, which is spatially white and circularly symmetric. As a result, the covariance matrix takes the following form:

$$\mathbf{R}_{\mathbf{x}_0} = \mathbf{A}(\theta)\mathbf{R}_s\mathbf{A}^H(\theta) + \sigma_n^2\mathbf{I} \quad (3)$$

where $\mathbf{R}_s$ and $\sigma_n$ are the signal source covariance matrix and the noise power, respectively.

## III. Fast Noise Subspace Estimation

It is shown in [3] that the columns of the reduced-dimensional transformation matrix $\mathbf{T}_D = [\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_D]$ of the MSWF are mutually orthogonal. Therefore, the rank $D$ MSWF is equivalent to solving the *Wiener Hopf* Equation $\mathbf{R}_{\mathbf{x}_0}\mathbf{w}_{wf} = \mathbf{r}_{\mathbf{x}_0 d_0}$ in the $D$-dimensional *Krylov* subspace $\mathcal{K}^{(D)}(\mathbf{R}_{\mathbf{x}_0}, \mathbf{r}_{\mathbf{x}_0 d_0}) = span\{\mathbf{r}_{\mathbf{x}_0 d_0}, \mathbf{R}_{\mathbf{x}_0}\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{R}_{\mathbf{x}_0}^{(D-1)}\mathbf{r}_{\mathbf{x}_0 d_0}\}$. It follows that $span\{\mathbf{t}_1, \cdots, \mathbf{t}_D\} = span\{\mathbf{r}_{\mathbf{x}_0 d_0}, \mathbf{R}_{\mathbf{x}_0}\mathbf{r}_{\mathbf{x}_0 d_0}, \cdots, \mathbf{R}_{\mathbf{x}_0}^{(D-1)}\mathbf{r}_{\mathbf{x}_0 d_0}\}$. Note that $\mathbf{R}_{\mathbf{x}_0}$ is Hermitian, thus the columns of $\mathbf{T}_D$ can be computed by the *Lanczos* algorithm. The recursive equation takes the following form

$$\mathbf{t}_k = \frac{\mathbf{R}_{\mathbf{x}_0}\mathbf{t}_{k-1} - \mathbf{t}_{k-2}^H\mathbf{R}_{\mathbf{x}_0}\mathbf{t}_{k-1}\mathbf{t}_{k-2} - \mathbf{t}_{k-1}^H\mathbf{R}_{\mathbf{x}_0}\mathbf{t}_{k-1}\mathbf{t}_{k-1}}{\|\mathbf{R}_{\mathbf{x}_0}\mathbf{t}_{k-1} - \mathbf{t}_{k-2}^H\mathbf{R}_{\mathbf{x}_0}\mathbf{t}_{k-1}\mathbf{t}_{k-2} - \mathbf{t}_{k-1}^H\mathbf{R}_{\mathbf{x}_0}\mathbf{t}_{k-1}\mathbf{t}_{k-1}\|_2} \quad (4)$$

where $\mathbf{P}_i = \mathbf{1} - \mathbf{t}_i\mathbf{t}_i^H$, $i \in \{k-1, k-2\}$. The *Lanczos* algorithm was recently employed in the MSWF by M. Joham *et al* [4].

It is worth noting that the columns of $\mathbf{T}_D$ are mutually orthogonal and contained in the true signal subspace, it follows that the column subspace of $\mathbf{T}_D$ is contained in the signal subspace, namely

$$S^D = span\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_D\} \subseteq S^P \quad (5)$$

Obviously, the column subspace of $\mathbf{T}_D$ is equivalent to the signal subspace if $D = P$. Thus, the redundant pre-filters of the MSWF span a noise subspace since all the redundant pre-filters after the $P$th stage are orthogonal to the signal subspace. However, the findings above do not hold in the case of coherent signals. A heuristic observation is that the pre-filter banks of the MSWF can be computed by *Lanczos* algorithm, and the *Lanczos* algorithm requires to estimate $\mathbf{R}_{\mathbf{x}_0}$. Hence, many techniques can be used to "de-correlate" the coherent signals. The *spatial smoothing* approach is applied herein for its simplicity in concept. It is shown that the spatially smoothed covariance matrix can be expressed as

$$\bar{\mathbf{R}}_{\mathbf{x}_0} = \frac{1}{M-m+1}\sum_{k=1}^{M-m+1}\mathbf{F}_k\mathbf{R}_{\mathbf{x}_0}\mathbf{F}_k^T \quad (6)$$

where $\mathbf{F}_k = [\ \mathbf{0}_{m\times(k-1)}\ \vdots\ \mathbf{I}_{m\times m}\ \vdots\ \mathbf{0}_{m\times(M-k-m+1)}\ ]$ is the selection matrix, $m$ is the number of elements of each subarray.

Consequently, the columns of $\mathbf{T}_D$ can be computed by the *Lanczos* algorithm since the spatially smoothed covariance matrix $\bar{\mathbf{R}}_{\mathbf{x}_0} \in \mathcal{C}^{m\times m}$ is the Hermitian matrix and $\|\ \bar{\mathbf{t}}_i\ \|_2 = 1$ holds for all $i \in \{1, 2, \cdots, D\}$. Hence, we get the recursion formula as follows

$$\begin{aligned}\bar{\mathbf{t}}_k &= \frac{\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1} - \bar{\mathbf{t}}_{k-2}^H\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1}\bar{\mathbf{t}}_{k-2} - \bar{\mathbf{t}}_{k-1}^H\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1}\bar{\mathbf{t}}_{k-1}}{\|\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1} - \bar{\mathbf{t}}_{k-2}^H\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1}\bar{\mathbf{t}}_{k-2} - \bar{\mathbf{t}}_{k-1}^H\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1}\bar{\mathbf{t}}_{k-1}\|_2} \\ &= \frac{\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1} - \gamma_{k-2,k-1}\bar{\mathbf{t}}_{k-2} - \gamma_{k-1,k-1}\bar{\mathbf{t}}_{k-1}}{\|\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_{k-1} - \gamma_{k-2,k-1}\bar{\mathbf{t}}_{k-2} - \gamma_{k-1,k-1}\bar{\mathbf{t}}_{k-1}\|_2}\end{aligned} \quad (7)$$

where $\gamma_{i,j} = \bar{\mathbf{t}}_i^H\bar{\mathbf{R}}_{\mathbf{x}_0}\bar{\mathbf{t}}_j$.

Suppose that there exists $L$ $(L > P)$ pre-filters of the MSWF such that the last $L-P$ columns of the redundant matrix $\Gamma_{L-D} = [\bar{\mathbf{t}}_{D+1}, \bar{\mathbf{t}}_{D+2}, \cdots, \bar{\mathbf{t}}_L]$ span a noise subspace, namely

$$\mathcal{N}^{L-P} = span\{\bar{\mathbf{t}}_{P+1}, \bar{\mathbf{t}}_{P+2}, \cdots, \bar{\mathbf{t}}_L\} \tag{8}$$

where $\mathcal{N}^{L-P}$ represents the $(L-P)$-dimensional noise subspace.

Since the spatially smoothed covariance matrix $\bar{\mathbf{R}}_{\mathbf{x}_0}$ is of rank $m$, the "spatially smoothed" observation vector should be $\bar{\mathbf{x}}_0(i) = \mathbf{x}_{0,1:m}(i)$, where $\mathbf{z}_{1:t}(i)$ denotes a vector formed by the the former $t$ elements of $\mathbf{z}(i)$. By substituting (6) into (7), the development of the *spatial smoothing Lanczos* (*SS-Lanczos*) algorithm can be fulfilled. The *SS-Lanczos* algorithm is summed as follows

Step1: $\bar{\mathbf{t}}_0 = 0, \quad \bar{\mathbf{t}}_1 = \frac{\mathbf{r}_{\bar{\mathbf{x}}_0 d_0}}{\|\mathbf{r}_{\bar{\mathbf{x}}_0 d_0}\|_2};$
$\gamma_{0,1} = 0, \quad \gamma_{1,1} = \bar{\mathbf{t}}_1^H \bar{\mathbf{R}}_{\mathbf{x}_0} \bar{\mathbf{t}}_1;$
$\Delta = 1;$
Step2: for $i = 2$ to M do
$\mathbf{v} = \bar{\mathbf{R}}_{\mathbf{x}_0} \bar{\mathbf{t}}_{i-1} - \gamma_{i-2,i-1} \bar{\mathbf{t}}_{i-2} - \gamma_{i-1,i-1} \bar{\mathbf{t}}_{i-1};$
$\gamma_{i-1,i} = \|\mathbf{v}\|_2;$
$\bar{\mathbf{t}}_i = \mathbf{v}/\gamma_{i-1,i};$
$\gamma_{i,i} = \bar{\mathbf{t}}_i^H \bar{\mathbf{R}}_{\mathbf{x}_0} \bar{\mathbf{t}}_i;$
if $|\bar{\mathbf{t}}_i^H \bar{\mathbf{t}}_1| \neq 0$ or i=L+1
then $\Delta = i - 1$ break;
end for
Step3: $\bar{\mathbf{T}}_D = [\bar{\mathbf{t}}_1, \bar{\mathbf{t}}_1, \cdots, \bar{\mathbf{t}}_D], \bar{\mathbf{\Gamma}}_{\Delta - D} = [\bar{\mathbf{t}}_{D+1}, \bar{\mathbf{t}}_{D+2}, \cdots, \bar{\mathbf{t}}_\Delta].$

Note that the algorithm above requires to estimate the covariance matrix $\bar{\mathbf{R}}_{\mathbf{x}_0}$, this leads to lower the mutually orthogonal property between the columns of $\bar{\mathbf{T}}_D$ since the samples is finite anywhere. Thus, the property can be used to stop the *SS-Lanczos* algorithm.

Once the noise subspace is acquired by the *SS-Lamczos* method, the MUSIC algorithm can be used to produce peaks at the true DOA locations of signals since the estimator only needs a simple one-dimensional search. Note that the *SS-Lamczos* estimator described in the paper merely requires $O(L^2MN)$ complex products operations [5]. However, the classical MUSIC method resorts to the eigendecomposition of the covariance matrix, which is of $O(M^3)$ operations. Thus the computational complexity of the new algorithm is greatly reduced.

## IV. Numerical Results

The receiving array herein is assumed to be a ULA with 32 isotropic sensors, whose spacings equal half-wavelength. Suppose that there are three groups impinging upon the array with there signals in the first group, two in the second and the third, respectively. Each group contains a direct-path signal and several scaled and delayed replicas of the direct-path signal that represent the multipaths or the "smart" jammers. The propagation constants of the three groups are $\{1, -0.8 + j0.6, -0.3 - j0.7\}$, $\{1, 0.5 + j0.7, \}$ and $\{1, 0.4 + j0.9\}$, respectively. In the sequel, the true DOAs are assumed to be $\{-9^0, 0^0, 24^0, 9^0, 19^0, -15^0\}$.

Fig. 1 and Fig. 2 show the spatial spectra of the MUSIC estimator based on the spatial smoothed *Lanczos* algorithm (SSL-MUSIC) and the spatial smoothed MUSIC method (SS-MUSIC), respectively. The number of snapshots is 256, the rank of the MSWF is 10. SNR equals 0dB. It is shown in Fig. 1 and Fig. 2 that the SSL-MUSIC method has the same resolution and estimation precision as the SS-MUSIC algorithm. However, the computational burden of the SSL-MUSIC technique is only of 691200 flops while the latter requires a complexity of 5038848 flops.

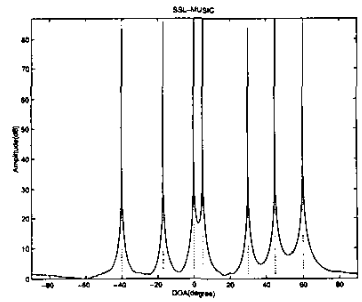


Fig. 1. SSL-MUSIC Spectrum.

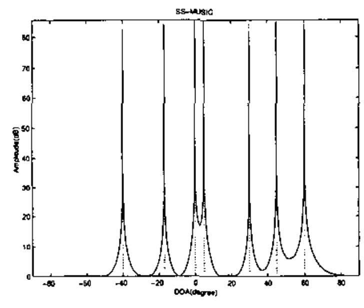


Fig. 2. SS-MUSIC Spectrum.

## V. Conclusion

A novel technique named spatial smoothing *Lanczos* (*SS-Lanczos*) algorithm is presented in this paper to estimate the noise subspace when coherent signals exist. The proposed method does not resort to the computation of the covariance matrix, its eigendecomposition and the backward recursion of the MSWF. Hence the computational complexity is dramatically reduced. Numerical results show that the proposed estimator has nearly the same resolution and precision as its counterparts based on the eigendecomposition.

## References


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