A Low-Complexity Nyström-Based Algorithm for Array Subspace Estimation

Cheng Qian
School of Electrical and
Information Engineering
Harbin Institute of Technology
Shenzhen Graduate School
Shenzhen, China 518055
Email: alextoqc@gmail.com

Lei Huang
School of Electrical and
Information Engineering
Harbin Institute of Technology
Shenzhen Graduate School
Shenzhen, China 518055
Email: lhuang8sasp@hotmail.com

***Abstract*—Subspace-based methods rely on singular value decomposition (SVD) of the sample covariance matrix (SCM) to compute the array signal or noise subspace. For large array, triditional subspace-based algorithms inevitably lead to intensive computational complexity due to both calculating SCM and performing SVD of SCM. To circumvent this problem, a Nyström-Based algorithm for array subspace estimation is proposed in this paper. In the proposed algorithm, we construct an approximated rank-k SVD of SCM without computing SCM, leading to computational simplicity. Statistical analysis and simulation results show that the rank-k SVD signal-subspace estimation algorithm (RKSSE) is computationally simple.**

I. Introduction

Large sensor arrays are widely used in detection and signal processing areas, such as direction of arrival[1], source localization[2], beamforming[3], biomedicine[4], etc.. And most of these algorithms are subspace based. However, as a result of the large-scale arrays which has a tremendous number of sensors, the use of subspace-based algorithms are hindered, because these methods are all computationally burdened since they inevitably involve the spectral decomposition of a covariance matrix to obtain eigenvectors and the corresponding eigenvalues.

In order to reduce the computational cost associated with subspace-based algorithms, we can estimate the SCM approximately[5] and develop some modefied low-complexity SVD or EVD method. For the first reason, Nicholas proposed a new covariance estimator based on the Nyström method for large array. For the second reason, a numerous alternatives were proposed by several authors. Marcos[6] proposed the propagator method to estimate noise subspace. Xin J.[7] used the least-mean-square (LMS) or normalized LMS (NLMS) algorithm to obtain signal-subspace. Recently, K. Mahata[8] used an alternative data model (RDM) to reduce the dimension of the signal subspace. Kaushik[9] developed weighted subspace fitting approaches using a modefied RDM and also proposed a computationally efficient suboptimal weighting method.

In this paper, the proposed method is expected to ensure computational savings and robustness. We begin with formulating the SCM and SVD problem in large array scenario and reviewing the Nyström method. Then we construct a rank-k SVD approach that keeps k largest eigenvalues of SCM and set the others to zeros to find the signal-subspace directly without applying SVD of SCM (SCM-SVD). At the end of the paper, an application to DOA estimation is presented through comparing RKSSE based MUSIC with SCM-SVD based MUSIC.

II. Problem Formulation

A. Signal Model

Consider a uniform linear array(ULA) with m identical sensors. There are $k(k < m)$ uncorrelated narrow-band source signals impinging on the array from directions $\theta_1, \theta_2, \cdots, \theta_k$ (θ is the azimuth angle). Assume that there are n snapshots $\mathbf{x}(1), \mathbf{x}(2), \cdots, \mathbf{x}(n)$ available. The $m \times 1$ array observation vector is modeled as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{s}(t)$ is $k \times 1$ vector of source waveforms; $\mathbf{n}(t)$ represents additive Gaussian noise with mean zero and autocorrelation matrix $\mathbb{E}\{\mathbf{n}(t)\mathbf{n}(t)^H\} = \sigma^2\mathbf{I}$ ($\mathbb{E}\{\cdot\}$ means statistical expection); $\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_k)] \in \mathbb{C}^{m \times k}$ denotes the array manifold matrix and

$$\mathbf{a}(\theta) = [1, e^{j(2\pi/\lambda)d\sin\theta}, \cdots, e^{j(2\pi/\lambda)d(m-1)\sin\theta}]^T$$

is the $m \times 1$ steering vector. In addition, λ is the carrier wavelength, d is the interelement spacing. The sample covariance matrix can be expressed as

$$\hat{\mathbf{R}} = \frac{1}{n}\mathbf{X}\mathbf{X}^H \tag{2}$$

where $\mathbf{X} = [\mathbf{x}(1), \mathbf{x}(2), \cdots, \mathbf{x}(n)]$ is the $m \times n$ data matrix. The superscripts ”T” and ”H” stand for the transpose and conjugate transpose, respectively.

B. Nyström Method For Matrix Approximation

Let $\mathbf{M} \in \mathbb{C}^{m \times m}$ be a aquare matrix. We decompose $\mathbf{M}$ as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \tag{3}$$

where $\mathbf{M}_{11} \in \mathbb{C}^{k\times k}$, $\mathbf{M}_{12} \in \mathbb{C}^{k\times(m-k)}$, $\mathbf{M}_{21} \in \mathbb{C}^{(m-k)\times k}$ and $\mathbf{M}_{22} \in \mathbb{C}^{(m-k)\times(m-k)}$. Let $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$ be the EVD of $\mathbf{M}_{11}$, where $\mathbf{U} \in \mathbb{C}^{k\times k}$ is the eigenvectors matrix and $\mathbf{\Lambda} \in \mathbb{C}^{k\times k}$ is the eigenvalues matrix. Our aim is to capture the colum eigenvectors of $\mathbf{M}$ according to $\mathbf{U}$. Here we define

$$\tilde{\mathbf{U}} = \mathbf{M}_{21}\mathbf{U}\mathbf{\Lambda}^{-1} \tag{4}$$

and

$$\tilde{\mathbf{V}} = \mathbf{\Lambda}^{-1}\mathbf{U}^{-1}\mathbf{M}_{12}. \tag{5}$$

Then we extend (4) and (5) into a matrix $\hat{\mathbf{U}}$, $\hat{\mathbf{V}}$, respectively, as the following form

$$\hat{\mathbf{U}} = \left[\begin{array}{c} \mathbf{U} \\ \mathbf{M}_{21}\mathbf{U}\mathbf{\Lambda} \end{array} \right] \tag{6}$$

and

$$\hat{\mathbf{V}} = \left[\begin{array}{cc} \mathbf{U}^{-1} & \mathbf{\Lambda}^{-1}\mathbf{U}^{-1}\mathbf{M}_{12} \end{array} \right] \tag{7}$$

The "Nyström" representation of $\hat{\mathbf{M}}$ becomes

$$\begin{aligned} \hat{\mathbf{M}} = \hat{\mathbf{U}}\mathbf{\Lambda}\hat{\mathbf{V}} &= \left[\begin{array}{c} \mathbf{U} \\ \mathbf{M}_{21}\mathbf{U}\mathbf{\Lambda}^{-1} \end{array} \right] \mathbf{\Lambda} \left[\begin{array}{cc} \mathbf{U}^{-1} & \mathbf{\Lambda}^{-1}\mathbf{U}^{-1}\mathbf{M}_{12} \end{array} \right] \\ &= \left[\begin{array}{cc} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{21}\mathbf{M}_{11}^{\dagger}\mathbf{M}_{12} \end{array} \right] \end{aligned} \tag{8}$$

where $(\cdot)^{\dagger}$ denotes the pseudo-inverse. Obviously, the Nyström approximation does not modify $\mathbf{M}_{11}$, $\mathbf{M}_{12}$ and $\mathbf{M}_{21}$, but replaces $\mathbf{M}_{22}$ by $\mathbf{M}_{21}\mathbf{M}_{11}^{\dagger}\mathbf{M}_{12}$.

III. Proposed Nyström-Based Rank-k SVD Signal-subspace Estimation Method

The covariance matrix $\mathbf{R} = \mathbb{E}\{\mathbf{X}\mathbf{X}^H\}$ is a symmetric matrix, so it can be partitioned as

$$\mathbf{R} = \left[\begin{array}{cc} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^H & \mathbf{R}_{22} \end{array} \right] \tag{9}$$

$\mathbf{X}$ is a $m \times n$ array received data matrix, partitioned as[5]

$$\mathbf{X} = \left[\begin{array}{c} \mathbf{X}_1 \\ \mathbf{X}_2 \end{array} \right] \tag{10}$$

where $\mathbf{X}_1 \in \mathbb{C}^{k\times n}$, $\mathbf{X}_2 \in \mathbb{C}^{(m-k)\times n}$ and k is the number of source signals. We define

$$\begin{aligned} \mathbf{R}_{11} &= \mathbb{E}\{\mathbf{X}_1\mathbf{X}_1^H\} \\ \mathbf{R}_{12} &= \mathbb{E}\{\mathbf{X}_1\mathbf{X}_2^H\} \\ \mathbf{R}_{22} &= \mathbb{E}\{\mathbf{X}_2\mathbf{X}_2^H\}. \end{aligned}$$

Our goal is to find a low-complexity method to approximate the eigenvalues and eigenvectors of the covariance matrix. Here we assume all the transmitted signals are uncorrelated, hence, $\mathbf{R}_{11}$ is a nonsingular matrix and $\mathrm{rank}(\mathbf{R}_{11}) = k$.

The apprximation technique is based on the Nyström method, through construct the SVD[10] of $\mathbf{R}_{\mathrm{NCE}}$ to approximate signal subspace, where $\mathbf{R}_{\mathrm{NCE}}$ is the Nyström-based covariance matrix estimator[5].
Let

$$\mathbf{F} = \left[\begin{array}{c} \mathbf{R}_{11} \\ \mathbf{R}_{12}^H \end{array} \right] \mathbf{R}_{11}^{-1/2},$$

the EVD of symmetric matrix $\mathbf{F}^H\mathbf{F}$ is given by $\mathbf{U}_{\mathrm{F}}\mathbf{\Lambda}_{\mathbf{F}}\mathbf{U}_{\mathbf{F}}^H$. Let

$$\mathbf{D} = \mathbf{\Lambda}_{\mathbf{F}}^{1/2}\mathbf{U}_{\mathbf{F}}^H\mathbf{U}_{\mathrm{F}}\mathbf{\Lambda}_{\mathbf{F}}^{1/2}$$

and the EVD of $\mathbf{D}$ is $\mathbf{U}_{\mathbf{D}}\mathbf{\Lambda}_{\mathbf{D}}\mathbf{U}_{\mathbf{D}}^H$, then the signal-subspace $\mathbf{U}_s \in \mathbb{C}^{m\times k}$ is given as

$$\mathbf{U}_s = \mathbf{F}\mathbf{U}_{\mathbf{F}}\mathbf{\Lambda}_{\mathbf{F}}^{-1/2}\mathbf{U}_{\mathbf{D}}. \tag{11}$$

We want to underline that the Nyström-based covariance estimator

$$\begin{aligned} \mathbf{R}_{\mathrm{NCE}} &= \mathbf{U}_s\Lambda_{\mathbf{F}}\mathbf{U}_s^H \\ &= \left[\begin{array}{cc} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^H & \mathbf{R}_{12}^H\mathbf{R}_{11}^{-1}\mathbf{R}_{12} \end{array} \right] \end{aligned} \tag{12}$$

IV. Simulation And Performance Analysis

A. Signal Subspace Deviation

Since the space spanned by the signal subspace is equivalent to the space spanned by the array manifold $\mathbf{A}(\theta)$[11], we use the function

$$\mathrm{D}(\hat{\mathbf{U}}_s, \mathbf{A}(\theta)\mathbf{T}) = \left\| \hat{\mathbf{U}}_s - \mathbf{A}(\theta)\mathbf{T} \right\|_F^2 \tag{13}$$

to define the deviation between the estimated signal subspace $\hat{\mathbf{U}}_s$ and the theoretical signal subspace.

Fig.1 shows the deviation of RKSSE and SCM-SVD method versus SNR. When the SNR is low, the performanc of the proposed method is a little poor than SCM-SVD. But with the increasing of SNR, RKSSE can achieve the same performance of SCM-SVD under a much lower computational complexity.

B. Computational Complexity

The computational complexity of SCM-SVD method is $O(m^3) + O(m^2 n)$ and mainly reflected in estimating SCM and do SVD of SCM. So this method on efficiency is intolerable for large array scenario. To the proposed method, however, it is with no need for computing the SCM and doing SVD. The computational bottleneck of the algorithm is in the formation of $\mathbf{F}^H\mathbf{F}$ and $\mathbf{U}_s$ with complexity $O(mk^2)$ and $O(mk + 2k^3)$, respectively. So the total computation load is about $O(mk^2 + mk)$ since $k \ll m$ in the large array application.

In the simulation, m varies from 20 to 400, snapshot is fixed to 100 and 4 uncorrelated signals impinging on the array. The simulation results depicted in Fig.2 provide a comparison of the computational complexity as a function of the number of sensor m. We can find the RKSSE algorithm is nearly linear increasing with m becoming larger for fixed k. When m is very large, RKSSE is much more efficient than the conventional method since RKSSE is insensitive of m.

C. Example: DOA Estimation

In this part, we consider a narrowband case where $(m, k) = (60, 4)$ and snapshot is fixed to 100. The true DOAs are $[-21^\circ, 2^\circ, 25^\circ, 50^\circ]$. The sensor array is linear uniform with $d/\lambda = 1/2$. The spatial noise $\mathbf{n}(t)$ is zero-mean, uniformly white Gaussian noise. SNR varies from 2 to 28dB to observe the performance of algorithms.

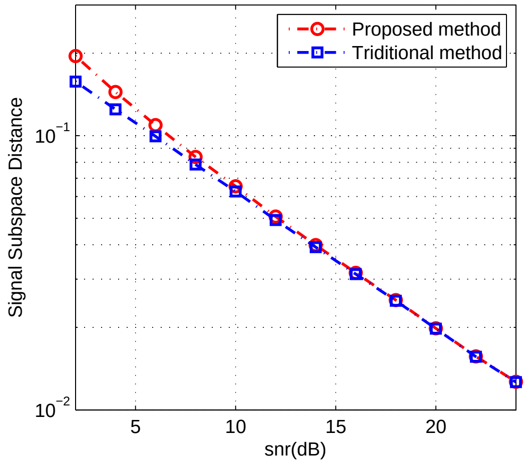

Fig. 1: Simulation of signal subspace deviation versus SNR. ($m = 60$, $k = 4$, $n = 100$, $SNR = 2 : 24$)

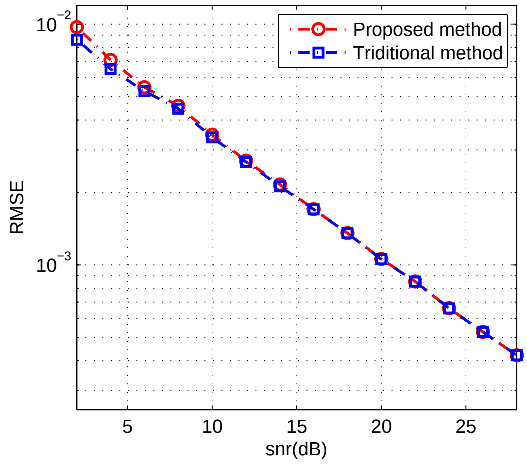

Fig. 3: RMSE angle error performance of the various algorithms in narrowband case versus SNR.

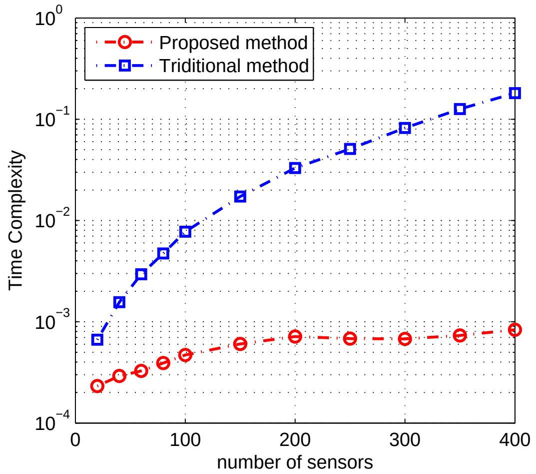

Fig. 2: Time complexity of the various algorithms versus m.

A Monte Carlo simulation was carried out to evalue the RMSE angle error performance of RKSSE and SCM-SVD based root-MUSIC. The number of Monte Carlo trials K is 500. RMSE is calculated by

$$\mathrm{RMSE} = \sqrt{\frac{1}{K}\sum_{k=1}^{K} \mid \hat{\theta}_k - \theta_k \mid^2}$$

In Fig.3, we can obtain that the proposed method achieves almost the same accuracy against SCM-SVD based when SNR is larger than 6dB.

V. Conclusion

In this article, we has addressed a Nyström-Based signal-subspace estimation method. Because of the computational advantages of the Nyström-based covariance estimator, its SVD can be fast constructed by a rank-k SVD approach which keeps k large eigenvalues of SCM and set the others to zeros. Through its use in DOA estimation, we can found the proposed method can substantially reduce computation with little decline in low SNR compared classical method.

References

[1] X. Mestre, M.A. Lagunas, "Modified Subspace Algorithms for DoA Estimation With Large Arrays," *IEEE Transactions on Signal Processing*, vol.56, no.2, pp.598-614, Feb. 2008.

[2] A. Manikas, Y.I. Kamil and M. Willerton, "Source Localization using Large Aperture Sparse Arrays," *IEEE Transactions on Signal Processing*, vol.PP, no.99, pp.1, 2012.

[3] P. Viswanath, D.N.C. Tse, R. Laroia, "Opportunistic beamforming using dumb antennas," *IEEE Transactions on Information Theory*, vol.48, no.6, pp.1277-1294, Jun 2002.

[4] E.J. Bond, et. cl, "Microwave imaging via space-time beamforming for early detection of breast cancer," *IEEE Transactions on Antennas and Propagation*, vol.51, no.8, pp. 1690- 1705, Aug. 2003.

[5] A. Nicholas and W.J. Patrick, "Estimating principal components of large covariance matrices using the Nyström method," *IEEE International Conference on Acoustics, Speech and Signal Processing(ICASSP)*, pp.3784-3787, 22-27, May 2011.

[6] S. Marcos, A. Marsal and M. Benidir, "The propagator method for source bearing estimation," *Signal Process*, vol.42, pp. 121C 138, 1995.

[7] Jingmin Xin; A. Sano, "Efficient subspace-based algorithm for adaptive bearing estimation and tracking," *IEEE Transactions on Signal Processing*, vol.53, no.12, pp. 4485- 4505, Dec. 2005.

[8] K. Mahata and T. Söderström, Esprit like estimation of real-valued sinusoidal frequencies, *IEEE Transactions on Signal Processing*, vol. 52, no. 5, 2004.

[9] K. Mahata, "Subspace fitting approaches for frequency estimation using real-valued data," *IEEE Transactions on Signal Processing*, vol.53, no.8, pp. 3099- 3110, Aug. 2005.

[10] Charless Fowlkes, etc., Spectral Grouping Using the Nyström Method, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 26, no. 2, pp. 214-225, February, 2004.

[11] R.O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Transactions on Antennas and Propagation*, vol.34, no.3, pp. 276- 280, Mar 1986.